

Telleparallel Lagrange Geometry and a Unified Field Theory: Linearization of the Field Equations

M. I. Wanas[†], Nabil L. Youssef[‡] and A. M. Sid-Ahmed^{‡*}

[†] Department of Astronomy, Faculty of Science, Cairo University
CTP of the British University in Egypt (BUE)
miwanas@sci.cu.edu.eg, mwanas@cu.edu.eg

[‡] Department of Mathematics, Faculty of Science, Cairo University
nlyoussef@sci.cu.edu.eg, nlyoussef2003@yahoo.fr

[‡] Department of Mathematics, Faculty of Science, Cairo University
amrs@mail.eun.eg, amrsidahmed@gmail.com

Abstract. The present paper is a natural continuation of our previous paper: "Teleparallel Lagrange geometry and a unified field theory, Class. Quantum Grav., 27 (2010), 045005 (29pp)" [14]. In this paper, we apply a linearization scheme on the field equations obtained in [14]. Three important results under the linearization assumption are accomplished. First, the vertical fundamental geometric objects of the EAP-space loose their dependence on the positional argument x . Secondly, our linearized theory in the Cartan-type case coincides with the GFT in the first order of approximation. Finally, an approximate solution of the vertical field equations is obtained.

Keywords: Extended Absolute Parallelism geometry, Euler-Lagrange equations, Generalized Field Theory, Extended Teleparallel Unified field theory, Linearization, Cartan-type case.

PACS: 04.50-h, 12.10-g, 45.10.Na, 02.40.Hw, 02.40.Ma.

MSC: 53B40, 53B50, 53Z05, 83C22.

*The authors are members of the Egyptian Relativity Group (ERG): www.erg.net.eg

1. Motivation and introduction

The theory under consideration in the present work is an *expansion* of the generalized field theory (GFT) [3] (formulated in the context of Absolute Parallelism (AP-) geometry (cf. [2], [12], [15]) to the *tangent bundle* TM . The theory is formulated in the context of Extended Absolute Parallelism (EAP-) geometry [16]. The EAP geometry, combines within its structure, the geometric richness of the tangent bundle (cf. [7]) and the mathematical simplicity of AP-geometry (cf. [9], [12]). The theory, which we refer to as the Extended Teleparallel Unified Field Theory (ETUFT), is constructed in a much wider and richer context than the GFT. Accordingly, the suggested theory has at least the advantages and features of its mother theory (and hopefully more). The GFT has the main properties required for any physical theory unifying gravity and electromagnetism:

- (1) Its material contents is totally induced by geometry [3].
- (2) All physical (and geometric) quantities of the theory are derived from one entity, namely, the building blocks of the geometry (the fundamental vector fields forming the parallelization) [3].
- (3) The theory shows that the charge of gravity (mass/energy) can generate electromagnetism [1], [5].
- (4) The theory shows that there is a direct relation between mass and electric charge. In particular, it shows that a part of mass is electromagnetic in origin and that the mass of the electron is totally electromagnetic in origin [13].

In addition to the above advantages enjoyed by the GFT, the ETUFT carries within its structure the potentiality of describing interactions other than gravity and electromagnetism [14]. In fact, we conjecture that the vertical field equations may express some kind of micro-(or quantum) properties.

For the above mentioned reasons and more, we are motivated to study the linearized form of the suggested field equations in the context of the ETUFT.

In the context of geometric field theories, the linearization scheme, although not covariant, has many advantages. Among these advantages is the determination of the constants or/and the parameters characterizing a certain theory. Another advantage is to test whether a nonlinear theory covers the domain of a previous, partially successful, linear theory. A third (and important) one is the attribution of some physical meaning to the geometric objects used to construct the theory.

The linearization scheme depends mainly on expanding different geometric objects, used in the construction of the theory, in terms of some small parameters and then neglecting terms of the second and higher order in these parameters. The neglect of such small quantities in the field equations implies the physical assumption that the field is weak. Also, the neglect of similar quantities in the equations of motion reflects the physical assumption that the motion is slow. The two assumptions characterize low energy systems. This provides us with a tool to test the theory in low energy regime.

It should be noted that the production of high energies to test some theories is difficult and sometimes even impossible because of both technological and budgetary reasons.

In the present work, we are going to linearize the field equation of the ETUFT [14]. In addition to the advantages of the linearization scheme mentioned above, we hope to throw more light on both the horizontal and vertical geometric objects used in the construction of the ETUFT. In other words, we hope to illuminate the role of the extra degrees of freedom implied by the ETUFT. A further advantage, which may be gained from linearization, is whether *one can explore the physical role of the nonlinear connection characterizing the underlying geometry*. Though the mathematical role of the nonlinear connection in the derived field equations is clear¹, the physical aspect of this nonlinear connection needs further investigation.²

The paper is organized in the following manner. In section 2, a short survey of the field equations obtained in the EAP-context is given. In section 3, we give a brief account on the process of linearization in the the classical AP-context followed by the process of linearization in the EAP-context. In section 4, we compute the fundamental tensor fields of the EAP-space under our linearization assumption. In section 5, we study the Cartan-type case under the linearized condition. In section 6, we give an approximate solution of the vertical field equations, and finally, we end the paper by some concluding remarks.

It should be noted that this paper is a (natural) continuation of [14]. Accordingly, we will use the results of [14] and stick to its notations.

¹In fact, the splitting of the field equations into horizontal and vertical counterparts is made possible due to the existence of the nonlinear connection.

²This may be partially achieved if an appropriate physical interpretation of the directional argument y is given.

2. A short survey of the field equations in the EAP-context

We first recall the fundamental tensor fields of the EAP-space. These are given in the following table [14].

Table 1: Fundamental second rank tensors of EAP-space

Horizontal		Vertical	
Skew-Symmetric	Symmetric	Skew-Symmetric	Symmetric
$\xi_{\mu\nu} := \gamma_{\mu\nu}^\alpha _\alpha$		$\xi_{ab} := \gamma_{ab}^d _d$	
$\gamma_{\mu\nu} := C_\alpha \gamma_{\mu\nu}^\alpha$		$\gamma_{ab} := C_d \gamma_{ab}^d$	
$\eta_{\mu\nu} := C_\beta \Lambda_{\mu\nu}^\beta$	$\phi_{\mu\nu} := C_\beta \Omega_{\mu\nu}^\beta$	$\eta_{ab} := C_d T_{ab}^d$	$\phi_{ab} := C_d \Omega_{ab}^d$
$\chi_{\mu\nu} := \Lambda_{\mu\nu}^\alpha _\alpha$	$\psi_{\mu\nu} := \Omega_{\mu\nu}^\beta _\beta$	$\chi_{ab} := T_{ab}^d _d$	$\psi_{ab} := \Omega_{ab}^d _d$
$\epsilon_{\mu\nu} := C_{\mu \nu} - C_{\nu \mu}$	$\theta_{\mu\nu} := C_{\mu \nu} + C_{\nu \mu}$	$\epsilon_{ab} := C_{a b} - C_{b a}$	$\theta_{ab} := C_{a b} + C_{b a}$
$U_{\mu\nu} := \gamma_{\alpha\mu}^\beta \gamma_{\nu\beta}^\alpha - \gamma_{\mu\alpha}^\beta \gamma_{\beta\nu}^\alpha$	$h_{\mu\nu} := \gamma_{\alpha\mu}^\beta \gamma_{\nu\beta}^\alpha + \gamma_{\mu\alpha}^\beta \gamma_{\beta\nu}^\alpha$	$U_{ab} := \gamma_{da}^c \gamma_{bc}^d - \gamma_{ad}^c \gamma_{cb}^d$	$h_{ab} := \gamma_{da}^c \gamma_{bc}^d + \gamma_{ad}^c \gamma_{cb}^d$
	$\sigma_{\mu\nu} := \gamma_{\alpha\mu}^\beta \gamma_{\beta\nu}^\alpha$		$\sigma_{ab} := \gamma_{da}^c \gamma_{cb}^d$
	$\omega_{\mu\nu} := \gamma_{\mu\alpha}^\beta \gamma_{\nu\beta}^\alpha$		$\omega_{ab} := \gamma_{ad}^c \gamma_{bc}^d$
	$\alpha_{\mu\nu} := C_\mu C_\nu$		$\alpha_{ab} := C_a C_b$

We now give a short survey of the field equations obtained in [14]. We take for the horizontal field equations a Lagrangian similar in form (but not in content) to that used by Mikhail and Wanas in their construction of the GFT [3]. We also assume that the nonlinear connection is *independent of the horizontal counterparts of the fundamental vector fields forming the parallelization*.³

In view of the above, for **the horizontal field equations**, we start with the following scalar Lagrangian: Let

$$\mathcal{H} = |\lambda| g^{\mu\nu} H_{\mu\nu},$$

³This condition is actually satisfied under the Cartan-type condition [14].

where

$$H_{\mu\nu} := \Lambda_{\epsilon\mu}^\alpha \Lambda_{\alpha\nu}^\epsilon - C_\mu C_\nu. \quad (2.1)$$

The Euler-Lagrange equations [7] for this Lagrangian are given by

$$\frac{\delta \mathcal{H}}{\delta \lambda_\beta} := \frac{\partial \mathcal{H}}{\partial \lambda_\beta} - \frac{\partial}{\partial x^\gamma} \left(\frac{\partial \mathcal{H}}{\partial \lambda_{\beta,\gamma}} \right) - \frac{\partial}{\partial y^a} \left(\frac{\partial \mathcal{H}}{\partial \lambda_{\beta;a}} \right) = 0. \quad (2.2)$$

Setting

$$E_\sigma^\beta := \frac{1}{|\lambda|} \left(\frac{\delta \mathcal{H}}{\delta \lambda_\beta} \right) \lambda_\sigma, \quad (2.3)$$

the Euler-Lagrange equations (2.2) take the form

$$\begin{aligned} 0 = E_\sigma^\beta &= \delta_\sigma^\beta H - 2H_\sigma^\beta - 2C_\sigma C^\beta - 2\delta_\sigma^\beta C^\epsilon|_\epsilon + 2\delta_\sigma^\beta C^\epsilon C_\epsilon - 2C^\epsilon \Lambda_{\epsilon\sigma}^\beta \\ &\quad + 2g^{\alpha\beta} C_{\sigma|\alpha} - 2g^{\gamma\alpha} \Lambda_{\sigma\alpha|\gamma}^\beta - 2N_{\alpha;a}^a (\Lambda^\alpha{}_\sigma{}^\beta - \Lambda^\beta{}_\sigma{}^\alpha) \\ &\quad + 2C^\nu (\delta_\sigma^\beta N_{\nu;a}^a - \delta_\nu^\beta N_{\sigma;a}^a) + 2g^{\alpha\beta} \{ \mathfrak{S}_{\sigma,\alpha,\epsilon} C_{\alpha a}^\epsilon R_{\sigma\epsilon}^a \} \end{aligned} \quad (2.4)$$

Lowering the index β in (2.4) and renaming the indices, we get

$$\begin{aligned} 0 = E_{\mu\nu} &:= g_{\mu\nu} H - 2H_{\mu\nu} - 2C_\mu C_\nu - 2g_{\mu\nu} (C^\epsilon|_\epsilon - C^\epsilon C_\epsilon) - 2C^\epsilon \Lambda_{\mu\epsilon\nu} + 2C_{\nu|\mu} \\ &\quad - 2g^{\epsilon\alpha} \Lambda_{\mu\nu\alpha|\epsilon} - 2N_{\epsilon;a}^a (\Lambda_{\nu\mu}^\epsilon - \Lambda_{\mu\nu}^\epsilon) + 2g_{\mu\nu} C^\epsilon N_{\epsilon;a}^a - 2C_\mu N_{\nu;a}^a \\ &\quad + 2 \mathfrak{S}_{\mu,\nu,\epsilon} C_{\mu a}^\epsilon R_{\nu\epsilon}^a. \end{aligned} \quad (2.5)$$

This is the generalized horizontal field equations in the context of the EAP-geometry.

Considering the symmetric part of (2.5), denoting $N_\beta := N_{\beta;a}^a$, it is found that

$$\begin{aligned} 0 = E_{(\mu\nu)} &:= (g_{\mu\nu} \mathring{R} - 2\mathring{R}_{(\mu\nu)}) + g_{\mu\nu} (\sigma - h - Q) - 2(\sigma_{\mu\nu} - h_{\mu\nu} - Q_{(\mu\nu)}) \\ &\quad + N^\beta (\Lambda_{\mu\nu\beta} + \Lambda_{\nu\mu\beta}) + 2g_{\mu\nu} C^\beta N_\beta - (C_\mu N_\nu + C_\nu N_\mu), \end{aligned} \quad (2.6)$$

which represents the symmetric part of the generalized horizontal field equations.

Setting

$$M_{\mu\nu} := N^\beta \Lambda_{\mu\nu\beta}, \quad Z_{\mu\nu} := C_\mu N_\nu, \quad Z := g^{\mu\nu} Z_{\mu\nu}, \quad (2.7)$$

we conclude, by (2.6), that

$$\mathring{R}_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} \mathring{R} = T_{(\mu\nu)}; \quad (2.8)$$

$$T_{(\mu\nu)} := \frac{1}{2} g_{\mu\nu} (\sigma - h - Q + 2Z) - (\sigma_{\mu\nu} - h_{\mu\nu} - Q_{(\mu\nu)}) + \left(\frac{1}{2} N_\beta \Omega_{\mu\nu}^\beta - Z_{(\mu\nu)} \right). \quad (2.9)$$

$T_{(\mu\nu)}$ is interpreted as the generalized energy momentum tensor, constructed from the symmetric tensors $\sigma_{\mu\nu}$, $h_{\mu\nu}$, $N_\beta \Omega_{\mu\nu}^\beta$, $Q_{(\mu\nu)}$ and $Z_{(\mu\nu)}$.

Consequently, the horizontal Einstein tensor has the form

$$\begin{aligned}\mathring{J}_{\mu\nu} &:= \mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{\mathcal{R}} \\ &= \left\{ \frac{1}{2} g_{\mu\nu} (\sigma - h) + (h_{\mu\nu} - \sigma_{\mu\nu}) \right\} + \frac{1}{2} g_{\mu\nu} (2Z - Q) + \left(\frac{1}{2} N_\beta \Omega_{\mu\nu}^\beta - Z_{(\mu\nu)} + Q_{(\mu\nu)} \right) \\ &\quad + \frac{1}{2} \mathfrak{S}_{\mu,\nu,\alpha} \mathring{C}_{\mu\alpha}^\alpha R_{\nu\alpha}^a,\end{aligned}\tag{2.10}$$

which is subject to the identity

$$\mathring{J}^\mu{}_{\sigma|\mu} = R_{\sigma\mu}^a \mathring{P}_a^\mu + \frac{1}{2} R_{\alpha\mu}^a \mathring{P}^{\alpha\mu}{}_{\sigma a}.\tag{2.11}$$

On the other hand, considering the skew-symmetric part of equation (2.5), it is found that

$$0 = E_{[\mu\nu]} = 2\{(\gamma_{\mu\nu} - \epsilon_{\mu\nu} - \xi_{\mu\nu} + N_\beta \Lambda_{\mu\nu}^\beta) + (M_{[\mu\nu]} - Z_{[\mu\nu]})\} + 3\mathfrak{S}_{\mu,\nu,\epsilon} C_{\nu a}^\epsilon R_{\epsilon\mu}^a.\tag{2.12}$$

Let us define

$$\begin{aligned}F_{\mu\nu} &:= (\gamma_{\mu\nu} - \xi_{\mu\nu} + \eta_{\mu\nu} + N_\beta \Lambda_{\mu\nu}^\beta) + (M_{[\mu\nu]} - Z_{[\mu\nu]}) + \frac{3}{2} \mathfrak{S}_{\mu,\nu,\epsilon} C_{\nu a}^\epsilon R_{\epsilon\mu}^a \\ &= (\gamma_{\mu\nu} - \xi_{\mu\nu} + \eta_{\mu\nu}) + N^\beta (\gamma_{\mu\nu\beta} + \Lambda_{\beta\mu\nu}) + \left(\frac{1}{2} N^\beta \Lambda_{\beta\mu\nu} - Z_{[\mu\nu]} \right) \\ &\quad + \frac{3}{2} \mathfrak{S}_{\mu,\nu,\epsilon} C_{\nu a}^\epsilon R_{\epsilon\mu}^a,\end{aligned}\tag{2.13}$$

then we get from (2.12) and (2.13)

$$F_{\mu\nu} = \delta_\nu C_\mu - \delta_\mu C_\nu\tag{2.14}$$

and

$$\mathfrak{S}_{\mu,\nu,\sigma} F_{\mu\nu}{}^\sigma = -\mathfrak{S}_{\mu,\nu,\sigma} R_{\mu\nu}^a \partial_a C_\sigma.\tag{2.15}$$

Accordingly, if $F_{\mu\nu}$ is interpreted as the horizontal electromagnetic field, then (2.15) represents the generalized horizontal Maxwell's equations and, in view of (2.14), C_μ is the horizontal electromagnetic potential. Again, by (2.13), $F_{\mu\nu}$ is constructed from the horizontal skew-symmetric fundamental tensors of the EAP-space (Table 1) together with the skew-symmetric tensors $N^\beta \gamma_{\mu\nu\beta}$, $N^\beta \Lambda_{\beta\mu\nu}$, $\mathfrak{S}_{\mu,\nu,\epsilon} C_{\nu a}^\epsilon R_{\epsilon\mu}^a$ and $Z_{[\mu\nu]}$. It is thus constructed from a purely geometric standpoint.

Moreover, if

$$J^\mu := F^{\mu\nu}{}_{|\nu},\tag{2.16}$$

then it is deduced that

$$J^\mu{}_{|\mu} = \frac{1}{2} \{ F^{\epsilon\mu} (\mathring{R}_{\mu\epsilon} - \mathring{R}_{\epsilon\mu}) + R_{\mu\nu}^a F^{\mu\nu}{}_{||a} \}.\tag{2.17}$$

In the case where the nonlinear connection N_μ^α is integrable [7], (2.17) can be viewed as a generalized conservation law and J^μ as the generalized horizontal current density.

For **the vertical field equations**, we consider a scalar Lagrangian formed of vertical entities, namely,

$$\mathcal{V} := ||\lambda||g^{ab}V_{ab},$$

where

$$V_{ab} := T_{ea}^d T_{db}^e - C_a C_b. \quad (2.18)$$

The Euler-Lagrange equations reduce to

$$\frac{\partial \mathcal{V}}{\partial \lambda_b} - \frac{\partial}{\partial y^e} \left(\frac{\partial \mathcal{V}}{\partial \lambda_{b,e}} \right) = 0. \quad (2.19)$$

In this case, we obtain,

$$\begin{aligned} 0 = E_{ab} := & g_{ab}V - 2V_{ab} - 2g_{ab}(C_{||e}^e - C^e C_e) - 2C_a C_b - 2C^e T_{aeb} \\ & + 2C_{b||a} - 2g^{de} T_{abe||d}. \end{aligned} \quad (2.20)$$

Considering the symmetric part of (2.20), we get

$$0 = E_{(ab)} := (g_{ab}\overset{\circ}{S} - 2\overset{\circ}{S}_{ab}) + g_{ab}(\bar{\sigma} - \bar{h}) - 2(\sigma_{ab} - h_{ab}), \quad (2.21)$$

so that

$$\overset{\circ}{S}_{ab} - \frac{1}{2}h_{ab}\overset{\circ}{S} = T_{ab}, \quad (2.22)$$

$$T_{ab} := \frac{1}{2}g_{ab}(\bar{\sigma} - \bar{h}) - (\sigma_{ab} - h_{ab}), \quad (2.23)$$

where

$$T^a{}_{b||a} = 0. \quad (2.24)$$

Consequently, in view of (2.22) and (2.24), T_{ab} could be interpreted as the generalized vertical energy-momentum tensor, which is, according to (2.23), constructed from the vertical symmetric fundamental tensors of the EAP-space (Table 1).

Considering the skew-symmetric part of (2.20), we conclude that if

$$F_{ab} := (\gamma_{ab} - \xi_{ab} + \eta_{ab}), \quad (2.25)$$

then

$$F_{ab} = \dot{\partial}_b C_a - \dot{\partial}_a C_b. \quad (2.26)$$

F_{ab} is interpreted as the generalized vertical electromagnetic field and C_a as the generalized vertical electromagnetic potential. Moreover, F_{ab} satisfies the differential identity

$$\mathfrak{S}_{a,b,c} F_{ab||c} = 0. \quad (2.27)$$

It is clear, by (2.25), that F_{ab} is constructed from the vertical skew-symmetric tensors of Table 1.

Finally, if we set

$$J^a := F^{ab}{}_{||b} \quad (2.28)$$

then, similar to (2.17), J^a represents the generalized vertical current density and satisfies the conservation law

$$J^a \circ_a = 0. \quad (2.29)$$

3. Linearization scheme in the EAP-context

We first give a brief account of the process of linearization in the the classical AP-context [4]. The vector fields λ_μ in the Minkowski space of special relativity are given by

$$\lambda_\mu = \delta_\mu, \quad (3.1)$$

where $i \in \{1, \dots, 4\}$, $\mu \in \{1, \dots, 4\}$ and δ_μ is the Kronecker delta. To get a space which differs slightly from the flat space, it is assumed that

$$\lambda_\mu = \delta_\mu + \epsilon h_\mu \quad (3.2)$$

where $h_\mu \in C^\infty(M)^4$ represents **perturbation** terms and the parameter ϵ is assumed to be of small magnitude compared to unity. In this scheme, each geometric object G defined in the AP-space can be expressed in the form

$$G = \sum_{r=0}^p \epsilon^r G^{(r)} \quad (p \in \{1, 2, \dots\}), \quad (3.3)$$

where r denotes the power of ϵ and $G^{(r)}$ is the coefficient of ϵ^r . For example, using (3.2), the metric tensor $g_{\mu\nu}$ is found to be

$$\begin{aligned} g_{\mu\nu} &= \lambda_\mu \lambda_\nu \\ &= (\delta_\mu + \epsilon h_\mu)(\delta_\nu + \epsilon h_\nu) \\ &= \delta_{\mu\nu} + \epsilon(\delta_\mu h_\nu + \delta_\nu h_\mu) + \epsilon^2(h_\mu h_\nu) \end{aligned}$$

This linearization procedure was carried out by Mikhial and Wanas for the GFT ([4], [11]). The results obtained showed a perfect agreement of GFT, in the first order of approximation, with both general relativity and Maxwell's theories. In addition, the linearized theory led to the prediction of new features, namely, the existence of a mutual interaction between both gravitational and electromagnetic fields [4].

In the present paper, we carry out a linearization of the field equations obtained in [14]. We will use the same conventions usually followed by physicists in the current literature in which, for example, mesh and world indices are mixed. The treatment here is, therefore, somewhat less rigorous than our previous paper [14].

In analogy to the above linearization process, we assume, in the context of EAP-space, that

$$\lambda_\mu = \delta_\mu + \epsilon h_\mu(x) + \epsilon k_\mu(y), \quad \lambda_a = \delta_a + \epsilon h_a(x) + \epsilon k_a(y), \quad (3.4)$$

⁴the algebra of smooth functions on M

where $i \in \{1, \dots, 4\}$, $\mu \in \{1, \dots, 4\}$, $a \in \{1, \dots, 4\}$ and $h_\mu, k_\mu, h_a, k_a \in C^\infty(TM)$ represent horizontal and vertical perturbation terms. Here h_μ, h_a are functions of the positional argument x only whereas k_μ, k_a are functions of the directional argument y only. Moreover, the parameters ϵ and ε are assumed to be of small magnitude compared to unity: $O(\epsilon) \simeq O(\varepsilon) \ll 1$ so that all terms of order ϵ^2 , $\epsilon\varepsilon$, ε^2 or higher orders can be neglected. This means that we are dealing with a **weak field**. Finally, e will denote either ϵ or ε so that, for example, $O(e^2)$ will mean either $O(\epsilon^2)$, $O(\epsilon\varepsilon)$ or $O(\varepsilon^2)$.

We interpret x^1, x^2 and x^3 as space coordinates whereas x^4 is taken to be the time coordinate. On the other hand, the vector y^a is attached as an *internal* variable to each point x^μ . In this sense, y^a may be regarded as the spacetime fluctuation (the micro-internal freedom) associated to the point x^μ ([7], [8].) We will return to the interpretation of the directional argument y later on.

4. First order approximation

We now carry out the task of computing the fundamental tensors of the EAP-space under the linearization assumption (3.4). The vertical (resp. horizontal) counterpart is obtained under no condition (resp. in the Cartan type case). Though, we have dealt with many cases in our previous paper [14], it is *the Cartan type case that can lend itself to the process of linearization*. This is because the nonlinear connection and hence all geometric objects considered are expressed explicitly in terms of the fundamental vector fields. On the other hand, a linearization of the horizontal field equations in the Berwald type case gives nothing new, since the derived horizontal field equations in this case actually coincide with those of the GFT [14].⁵

In preparation to what follows, we set

$$z_{\mu\nu} := (h_\nu + h_\mu), \quad w_{\mu\nu} := (k_\nu + k_\mu); \quad h_\nu := \delta_\mu h_\nu, \quad k_\nu := \delta_\mu k_\nu \quad (4.1)$$

$$z_{ab} := (h_b + h_a), \quad w_{ab} := (k_b + k_a); \quad h_b := \delta_a h_b, \quad k_b := \delta_a k_b. \quad (4.2)$$

Then, in view of (3.4), we obtain

Theorem 4.1. *To the first order of approximation, we have*

- (a) $\lambda_i^\mu \simeq \delta_i^\mu - \epsilon h_i(x) - \varepsilon k_i(y); \quad \lambda_i^a \simeq \delta_i^a - \epsilon h_i(x) - \varepsilon k_i(y).$
- (b) $g_{\mu\nu} \simeq \delta_{\mu\nu} + \epsilon z_{\mu\nu}(x) + \varepsilon w_{\mu\nu}(y); \quad g^{\mu\nu} \simeq \delta_{\mu\nu} - \epsilon z_{\mu\nu}(x) - \varepsilon w_{\mu\nu}(y).$
- (c) $g_{ab} \simeq \delta_{ab} + \epsilon z_{ab}(x) + \varepsilon w_{ab}(y); \quad g^{ab} \simeq \delta_{ab} - \epsilon z_{ab}(x) - \varepsilon w_{ab}(y).$
- (d) $C_{bc}^a \simeq \varepsilon(k_{b;c})(y).$
- (e) $T_{bc}^a \simeq \varepsilon(k_{b;c} - k_{c;b})(y).$
- (f) $C_b \simeq \varepsilon(k_{b;a} - k_{a;b})(y).$

⁵This implies, in particular, that both Maxwell's and general relativity theories are an outcome of our field equations under the Berwald condition.

- (g) $\overset{\circ}{C}_{bc}^a \simeq \frac{1}{2} \varepsilon(w_{ab;c} + w_{ac;b} - w_{bc;a})(y)$.
- (h) $\gamma_{bc}^a \simeq \varepsilon\{k_{b;c} - \frac{1}{2}(w_{ab;c} + w_{ac;b} - w_{bc;a})\}(y)$.
- (j) $\Omega_{bc}^a \simeq \varepsilon\{(k_{b;c} + k_{c;b}) - (w_{ab;c} + w_{ac;b} - w_{bc;a})\}(y)$.

Proof. We prove (a), (e) and (g) only. The rest can be proved in a similar manner.

- (a) Assume, to the first order of e , that $\lambda_i^\mu = D_i^\mu + \epsilon H_i^\mu + \varepsilon K_i^\mu$. Then, in view of $\lambda_i^\mu \lambda_\nu^\mu = \delta_{i\nu}^\mu$, we have $D_i^\mu = \delta_{i\mu}$, $\epsilon\{H_i^\mu \delta_\nu^\mu + h_{i\nu} \delta_\mu^\mu\} = 0$ and $\varepsilon\{K_i^\mu \delta_\nu^\mu + k_{i\nu} \delta_\mu^\mu\} = 0$. By the second relation above, we get $H_i^\mu \delta_\nu^\mu = -h_{i\nu} \delta_\mu^\mu$. Multiplying by δ_ν^μ , we obtain $H_j^\mu = -(\delta_\nu^\mu h_{i\nu}) \delta_\mu^\mu = -h_{ij} \delta_\mu^\mu := -h_{ij}$. Similarly, by $\varepsilon\{K_i^\mu \delta_\nu^\mu + k_{i\nu} \delta_\mu^\mu\} = 0$, we conclude that $K_j^\mu = -k_{ij}$. The expression of λ_i^a is obtained in a similar manner, using the relation $\lambda_i^a \lambda_b^a = \delta_b^a$.

- (e) By (a), we have

$$C_{bc}^a = \lambda_i^a \partial_c \lambda_b^i = \{\delta_a^i - \epsilon h_i^a - \varepsilon k_i^a + O(e^2)\} \{\varepsilon k_{b;c}^i\} = \varepsilon k_{b;c}^a + O(e^2).$$

- (g) By (c), we have $g^{ad} = \delta_{ad} - \epsilon z_{ad}(x) - \varepsilon w_{ad}(y) + O(e^2)$. Moreover,

$$g_{cd;b} = \varepsilon w_{cd;b}(y) + O(e^2), \quad g_{bd;c} = \varepsilon w_{bd;c}(y) + O(e^2), \quad g_{bc;d} = \varepsilon w_{bc;d}(y) + O(e^2).$$

Consequently,

$$\begin{aligned} \overset{\circ}{C}_{bc}^a &= \frac{1}{2} g^{ad} \{g_{cd;b} + g_{bd;c} - g_{bc;d}\} \\ &\simeq \frac{1}{2} \varepsilon \delta_{ad} \{w_{dc;b} + w_{db;c} - w_{bc;d}\}(y) = \frac{1}{2} \varepsilon \{w_{ab;c} + w_{ac;b} - w_{bc;a}\}(y). \quad \square \end{aligned}$$

In the light of the above theorem, using the relevant definitions, we obtain

Proposition 4.2. *To the first order of ε , the following hold:*

- (a) $\epsilon_{ab} \simeq \varepsilon(k_{a;bd} - k_{b;ad})(y)$,
- (b) $\theta_{ab} \simeq \varepsilon(k_{a;bd} + k_{b;ad} - 2k_{d;ab})(y)$,
- (c) $\xi_{ab} \simeq \varepsilon\{k_{b;dd} - \frac{1}{2}(w_{ab;dd} + w_{ad;bd} - w_{bd;ad})\}(y)$,
- (d) $\psi_{ab} \simeq \varepsilon\{(k_{a;bd} + k_{b;ad}) - (w_{ad;bd} + w_{bd;ad} - w_{ab;dd})\}(y)$,
- (e) $\chi_{ab} \simeq \varepsilon(k_{a;bd} - k_{b;ad})(y)$.

Let $W_{abc} := \frac{1}{2}(w_{ab;c} + w_{ac;b} - w_{bc;a})$. Then the following tensors contain no terms of first order:

Proposition 4.3. *To the second order of ε , the following hold:*

- (a) $\gamma_{ab} \simeq \varepsilon^2\{(k_{d;c} - k_{c;d})(k_{b;d} - W_{abd})\}(y)$

- (b) $\phi_{ab} \simeq \varepsilon^2 \{ (k_{d;c} - k_{c;d})(k_{a;b} + k_{b;a} - 2W_{dab}) \}(y).$
- (c) $\eta_{ab} \simeq \varepsilon^2 \{ (k_{d;c} - k_{c;d})(k_{a;b} - h_{b;a}) \}(y).$
- (d) $\omega_{ab} \simeq \varepsilon^2 \{ (k_{d;c} - W_{dac})(k_{b;d} - W_{cbd}) \}(y).$
- (e) $\sigma_{ab} \simeq \varepsilon^2 \{ (k_{c;a} - W_{dca})(k_{d;b} - W_{cdb}) \}(y).$
- (f) $\alpha_{ab} \simeq \varepsilon^2 \{ (k_{a;c} - k_{c;a})(k_{b;c} - k_{c;b}) \}(y).$
- (g) $h_{ab} \simeq \varepsilon^2 \{ (k_{c;a} - W_{dca})(k_{b;d} - W_{cbd}) + (k_{a;c} - W_{dac})(k_{d;b} - W_{cdb}) \}(y).$
- (h) $U_{ab} \simeq \varepsilon^2 \{ (k_{c;a} - W_{dca})(k_{b;d} - W_{cbd}) - (k_{a;c} - W_{dac})(k_{d;b} - W_{cdb}) \}(y).$

Corollary 4.4. *Assume that (3.4) holds. Then, up to the first order of approximation, the vertical fundamental geometric objects of the EAP-space are functions of the directional argument y only.*

5. Linearization in the Cartan-type case

We now consider the Cartan-type case. In this case, the nonlinear connection coefficients N_μ^a are given by $N_\mu^a = y^b (\lambda_i^a \partial_\mu \lambda_b)$ [16]. Consequently, all geometric objects of the EAP-space are expressed explicitly in terms of the fundamental vector fields forming the parallelization.

Similar to Theorem 4.1, we have the following

Theorem 5.1. *Assume that the canonical d -connection is of Cartan type. Then, up to the first order approximation, the following hold:*

- (a) $N_\mu^a \simeq \epsilon \{ y^b (h_{b,\mu}(x)) \}.$
- (b) $\Gamma_{\mu\nu}^\alpha \simeq \epsilon (h_{\mu,\nu})(x), \quad \Gamma_{b\nu}^a \simeq \epsilon (h_{b,\nu})(x).$ Consequently, $N_\nu := N_{\nu;a}^a \simeq \epsilon h_{a,\nu}(x).$
- (c) $\Lambda_{\mu\nu}^\alpha \simeq \epsilon (h_{\mu,\nu} - h_{\nu,\mu})(x).$
- (d) $C_\mu \simeq \epsilon (h_{\mu,\alpha} - h_{\alpha,\mu})(x).$
- (e) $\mathring{\Gamma}_{\mu\nu}^\alpha \simeq \frac{1}{2} \epsilon (z_{\mu\alpha,\nu} + z_{\nu\alpha,\mu} - z_{\mu\nu,\alpha})(x).$
- (f) $\gamma_{\mu\nu}^\alpha \simeq \epsilon \{ h_{\mu,\nu} - \frac{1}{2} (z_{\mu\alpha,\nu} + z_{\nu\alpha,\mu} - z_{\mu\nu,\alpha}) \}(x).$
- (g) $\Omega_{\mu\nu}^\alpha \simeq \epsilon \{ (h_{\mu,\nu} + h_{\nu,\mu}) - (z_{\mu\alpha,\nu} + z_{\nu\alpha,\mu} - z_{\mu\nu,\alpha}) \}(x).$

Proof. We prove (a) only. The rest is similar. We have

$$\begin{aligned}
N_\mu^a &= y^b (\lambda_i^a \partial_\mu \lambda_b) \\
&= y^b \{ \delta_i^a - \epsilon h_i^a - \varepsilon k_i^a + O(e^2) \} \{ \epsilon h_{b,\mu} \} = \epsilon (y^b h_{b,\mu}^a) + O(e^2).
\end{aligned}$$

□

In the next two propositions, we assume that the canonical d -connection is of **Cartan type**. Then, using the relevant definitions, taking into account Theorem 5.1 and setting $Z_{\beta\mu\nu} := \frac{1}{2}(z_{\beta\mu,\nu} + z_{\beta\nu,\mu} - z_{\nu\mu,\beta})$, we obtain

Proposition 5.2. *To the first order of ϵ , we have*

- (a) $\epsilon_{\mu\nu} \simeq \epsilon(h_{\alpha\mu,\alpha\nu} - h_{\alpha\nu,\alpha\mu})(x)$,
- (b) $\theta_{\mu\nu} \simeq \epsilon(h_{\alpha\mu,\alpha\nu} + h_{\alpha\nu,\alpha\mu} - 2h_{\alpha\alpha,\mu\nu})(x)$,
- (c) $\xi_{\mu\nu} \simeq \epsilon\{h_{\mu,\alpha\alpha} - Z_{\mu\nu\alpha,\alpha}\}(x)$,
- (d) $\psi_{\mu\nu} \simeq \epsilon\{(h_{\alpha\mu,\nu\alpha} + h_{\alpha\nu,\mu\alpha}) - 2Z_{\alpha\mu\nu,\alpha}\}(x)$,
- (e) $\chi_{\mu\nu} \simeq \epsilon(h_{\alpha\mu,\nu\alpha} - h_{\alpha\nu,\mu\alpha})(x)$.

The following tensors are of order ϵ^2 .

Proposition 5.3. *To the second order of ϵ , we have*

- (a) $\gamma_{\mu\nu} \simeq \epsilon^2\{(h_{\alpha\alpha,\beta} - h_{\beta\beta,\alpha})(h_{\nu,\alpha} - Z_{\mu\nu\alpha})\}(x)$
- (b) $\phi_{\mu\nu} \simeq \epsilon^2\{(h_{\alpha\alpha,\beta} - h_{\beta\beta,\alpha})(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - 2Z_{\alpha\mu\nu})\}(x)$.
- (c) $\eta_{\mu\nu} \simeq \epsilon^2\{(h_{\alpha\alpha,\beta} - h_{\beta\beta,\alpha})(h_{\alpha\mu,\nu} - h_{\alpha\nu,\mu})\}(x)$.
- (d) $\omega_{\mu\nu} \simeq \epsilon^2\{(h_{\alpha\mu,\beta} - Z_{\alpha\mu\beta})(h_{\nu,\alpha} - Z_{\beta\nu\alpha})\}(x)$.
- (e) $\sigma_{\mu\nu} \simeq \epsilon^2\{(h_{\alpha\beta,\mu} - Z_{\alpha\beta\mu})(h_{\alpha\alpha,\nu} - Z_{\beta\alpha\nu})\}(x)$.
- (f) $\alpha_{\mu\nu} \simeq \epsilon^2\{(h_{\beta\mu,\beta} - h_{\beta\beta,\mu})(h_{\nu,\sigma} - h_{\sigma,\nu})\}(x)$.
- (g) $h_{\mu\nu} \simeq \epsilon^2\{(h_{\alpha\alpha,\mu} - Z_{\beta\alpha\mu})(h_{\nu,\beta} - Z_{\alpha\nu\beta}) + (h_{\beta\mu,\alpha} - Z_{\beta\mu\alpha})(h_{\alpha\beta,\nu} - Z_{\alpha\beta\nu})\}(x)$.
- (h) $U_{\mu\nu} \simeq \epsilon^2\{(h_{\alpha\alpha,\mu} - Z_{\beta\alpha\mu})(h_{\nu,\beta} - Z_{\alpha\nu\beta}) - (h_{\beta\mu,\alpha} - Z_{\beta\mu\alpha})(h_{\alpha\beta,\nu} - Z_{\alpha\beta\nu})\}(x)$.

The relation equivalent to (2.5) in the Cartan type case is given by

$$0 = E_{\mu\nu} := g_{\mu\nu}H - 2H_{\mu\nu} - 2C_\mu C_\nu - 2g_{\mu\nu}(C^\epsilon_{|\epsilon} - C^\epsilon C_\epsilon) - 2C^\epsilon \Lambda_{\mu\epsilon\nu} + 2C_{\nu|\mu} - 2g^{\epsilon\alpha} \Lambda_{\mu\nu\alpha|\epsilon} - 2N_\epsilon(\Lambda^\epsilon_{\nu\mu} - \Lambda_{\mu\nu}{}^\epsilon) + 2g_{\mu\nu}C^\epsilon N_\epsilon - 2C_\mu N_\nu. \quad (5.1)$$

Consequently, $C_\mu N_\nu$, $C^\mu N_\mu$, $N^\beta \Lambda_{\mu\nu\beta}$ and $N^\beta \Lambda_{\beta\mu\nu}$ are the additional terms appearing in the horizontal field equations under the Cartan condition that have no counterparts in the field equations obtained in the context of the GFT.

Proposition 5.4. *The following holds:*

- (a) $C_\mu N_\nu = \epsilon^2\{(h_{\alpha\mu,\alpha} - h_{\alpha\alpha,\mu})h_{c,\nu}\}(x)$.

$$(b) \quad N^\beta \Lambda_{\mu\nu\beta} = \epsilon^2 \{h_{c,\beta}(h_{\nu,\beta} - (h_{\beta,\nu}))\}(x).$$

$$(c) \quad N^\beta \Lambda_{\beta\mu\nu} = \epsilon^2 \{h_{c,\beta}(h_{\mu,\nu} - h_{\nu,\mu})\}(x).$$

In view of Propositions (5.2), (5.3) and (5.4), we have

Corollary 5.5. *Assume that (3.4) holds. Then, up to the first order of approximation, the purely horizontal fundamental geometric objects of the EAP-space in the Cartan type case are identical to their corresponding counterparts in the context of classical AP-space [11] and are functions of the positional argument x only. This is because all extra terms appearing in the horizontal field equations in ETUFT are of order ϵ^2 . Consequently, the horizontal field equations under the Cartan condition coincides with the GFT up to the first order of approximation.*

One reading of Corollary 5.5 is that our constructed unified field theory seems to be a plausible generalization of the GFT. Not only do the horizontal field equations in the Berwald-type case coincide with those of the GFT [14], but also *the horizontal field equations in the Cartan-type case coincide with the GFT in the first order of approximation*. This also means that our theory (under the Cartan type condition) *differs* from the GFT *only* when dealing with *strong* fields, that is, in the second (and higher orders) of ϵ . In other words, *the two theories actually coincide when dealing with weak fields*.

6. Approximate solutions of the vertical field equations

In this final section, we will examine the solutions of the vertical field equations corresponding to the first order of approximation.

By (2.22) and (2.23), the vertical Einstein tensor is expressed in terms of the fundamental tensors in the form:

$$\mathring{S}_{ab} - \frac{1}{2} h_{ab} \mathring{S} = \frac{1}{2} g_{ab} (\bar{\sigma} - \bar{h}) - (\sigma_{ab} - h_{ab}). \quad (6.1)$$

As easily checked, in view of Theorem 4.1 (c) and Proposition 4.3 (e) and (g), (6.1) reduces in the first order of approximation to

$$(\mathring{S}_{ab})^{(1)} - \frac{1}{2} \delta_{ab} \mathring{S}^{(1)} = 0. \quad (6.2)$$

Contracting, we get $\mathring{S}^{(1)} = 0$, so that (6.2) reduces to

$$(\mathring{S}_{ab})^{(1)} = 0. \quad (6.3)$$

Now, we have [14]

$$\mathring{S}_{ab} = -\frac{1}{2} (\theta_{ab} - \psi_{ab} + \phi_{ab}) + \omega_{ab}. \quad (6.4)$$

Consequently, in view of Proposition 4.3 ($(\phi_{ab})^{(1)} = (\omega_{ab})^{(1)} = 0$), (6.4) reduces in the first order of approximation to

$$(\mathring{S}_{ab})^{(1)} = \frac{1}{2} \{(\psi_{ab})^{(1)} - (\theta_{ab})^{(1)}\}, \quad (6.5)$$

so that, by Proposition 4.2 **(b)** and **(d)**, noting that $w_{dd} = 2k_d$, we obtain

$$\begin{aligned} (\mathring{S}_{ab})^{(1)} &= \frac{1}{2} \{ (k_{a;bd} + k_{b;ad}) - w_{da;bd} - w_{db;ad} + w_{ab;dd} \\ &\quad - (k_{a;db} + k_{b;ad}) + 2k_{d;ab} \} \\ &= \frac{1}{2} \{ w_{ab;dd} - w_{ad;bd} - w_{bd;ad} + w_{dd;ab} \}. \end{aligned}$$

The above equation, together with (6.3), gives

$$w_{ab;dd} - w_{ad;bd} - w_{bd;ad} - w_{dd;ab} = 0. \quad (6.6)$$

Consequently,

$$\begin{aligned} w_{ab;dd} &= w_{ad;bd} + w_{bd;ad} - w_{dd;ab} \\ &= (w_{ad;bd} - \frac{1}{2} w_{dd;ab}) + (w_{bd;ad} - \frac{1}{2} w_{dd;ab}), \end{aligned}$$

which can be expressed in the form

$$w_{ab;dd} = \{ (w_{ad;d} - \frac{1}{2} w_{dd;a})_{;b} + (w_{bd;d} - \frac{1}{2} w_{dd;b})_{;a} \}. \quad (6.7)$$

Hence, if $(\mathring{C}_{dd}^a)^{(1)} = 0$, that is, $w_{ad;d} = \frac{1}{2} w_{dd;a}$, then the symmetric part of the vertical field equations in this case reduces to

$$w_{ab;dd} = \{ \frac{\partial^2}{\partial(y^1)^2} + \frac{\partial^2}{\partial(y^2)^2} + \frac{\partial^2}{\partial(y^3)^2} + \frac{\partial^2}{\partial(y^4)^2} \} w_{ab} = 0. \quad (6.8)$$

In view of the interpretation of the variable y^a , (6.8) represents, *à la Kaluza*, a *wave equation in the micro-internal dimension*.

On the other hand, by (2.25) and (2.26), the vertical generalized electromagnetic field is given by

$$F_{ab} := (\gamma_{ab} - \xi_{ab} + \eta_{ab}), \quad (6.9)$$

$$F_{ab} = \dot{\partial}_b C_a - \dot{\partial}_a C_b. \quad (6.10)$$

By Proposition 4.3, (6.9) and (6.10) reduce in the first approximation to

$$(C_{a;b})^{(1)} - (C_{b;a})^{(1)} = -(\xi_{ba})^{(1)} \quad (6.11)$$

Now, in the first order of ε ,

$$C_a \simeq \varepsilon(k_{a;d} - k_{d;a}) \simeq \varepsilon(C_a)^{(1)}; \quad C_{a;b} \simeq \varepsilon(k_{a;db} - k_{d;ab}) \simeq \varepsilon(C_{a;b})^{(1)}. \quad (6.12)$$

Consequently,

$$(C_{a;b})^{(1)} = (C_a)^{(1)}_{;b} = k_{a;db} - k_{d;ab}. \quad (6.13)$$

Contraction of relation (6.6) yields

$$w_{dd;aa} = w_{ad;ad}, \quad (6.14)$$

equivalently,

$$k_{d;aa} = k_{a;da}. \quad (6.15)$$

Hence, in view of (6.13) and (6.15), we conclude that

$$(C_{d;d})^{(1)} = (C_d)^{(1)}{}_{;d} = 0 \quad (6.16)$$

Differentiating (6.11), taking into account (6.16), we obtain

$$(C_{a;dd})^{(1)} = (C_a)^{(1)}{}_{;dd} = -(\xi_{da})^{(1)}{}_{;d} = -(\xi_{da;d})^{(1)} \quad (6.17)$$

Consequently, by $J^a := F^{ab}{}_{;b}$, where J_a is the vertical current density, we obtain

$$\left\{ \frac{\partial^2}{\partial(y^1)^2} + \frac{\partial^2}{\partial(y^2)^2} + \frac{\partial^2}{\partial(y^3)^2} + \frac{\partial^2}{\partial(y^4)^2} \right\} (C_a)^{(1)} = -(J_a)^{(1)}. \quad (6.18)$$

Hence, if $(J_a)^{(1)} = 0$, then $(C_a)^{(1)}$ again satisfies a *wave equation in the micro-internal dimension*.

The vertical part of the field equations gives rise to two different 4-dimensional Laplace equations which would be wave equations if the metric was not positive definite, but had Lorentz signature. The meaning of this result, however, will be clarified if a clear *physical interpretation of the vector y^a is given*.

We end the paper with the following remarks and comments:

- As previously mentioned, a possible interpretation of the coordinates would to take x_1, x_2 and x_3 as space coordinates and x^4 as time coordinates. On the other hand, the vector y^a is attached as an internal variable to each x^μ . According to Miron [8], y^a may be regarded as spacetime fluctuations (micro internal freedom) associated to the point x^μ . It could be argued that this interpretation, however, seems somewhat incompatible with (3.4). This is because we are actually combining dimensions of different natures, namely, a **macro** dimension x and a **micro** dimension y . To avoid this (apparent) incompatibility, we shall take x^μ as stated above and *leave the meaning of the directional argument y^a unspecified, at least for the present*.
- In any geometric field theory, authors try to attribute physical meaning to the geometric objects present in the theory. One way to accomplish this is to compare the new theory with previously existing field theories. The GFT is a field theory unifying electromagnetism and gravitation. It is thus a theory that generalizes both Maxwell's electromagnetic theory and Einstein's general theory of relativity. The comparison of the GFT with these two theories, resulted in attributing some physical meaning to certain geometric objects occurring in the GFT (using certain systems of units). A new scheme, called **Type Analysis** [13], has also

been suggested in the context of the GFT. Its aim is to test the AP-geometry (the underlying geometry of the GFT) for representing certain physical fields. The procedure of Type Analysis is usually applied before solving the field equations. It is a covariant procedure formulated in the language of tensors. Some tensors are used to characterize the AP-space under consideration. Roughly, the types of spaces considered are as follows: spaces with or without electromagnetic field, spaces with or without gravitational fields, and, finally, spaces in which both fields are present. The strengths of the fields involved are also taken into account. More precisely, some tensors are defined indicating the presence or absence of electromagnetic and gravitational fields. Not all possible combinations, however, are allowed. There are certain constraints. For example, a space with a non-vanishing electromagnetic field necessarily contains a non-vanishing gravitational field.⁶

- One of our future aims is to generalize the procedure of Type Analysis to our constructed field theory. It should be noted that this is not an easy task. This is because both the physical and the mathematical scope of the ETUFT are much wider and richer in content than the GFT. Therefore, in the context of the ETUFT, an “extension” of the procedure of Type Analysis (applied to the (horizontal) field equations in the Cartan type case and the vertical field equations in the general case) is expected to be more complicated and much more involved. Some of the reasons for such complications are the following: First, the (horizontal) field equations in the Cartan type are more complex than their corresponding counterparts in the GFT (they coincide only in the first order of approximation). Secondly, we have two types of “e” in our linearized field theory. Last, but not least, our geometric objects, unlike the classical AP-geometry (in which the GFT is formulated), depend, in general, on both the directional argument x and the positional argument y .
- On the other hand, an application of (some method similar to) the procedure of Type analysis to the ETUFT may help us illuminate the physical role of the nonlinear connection and give a plausible physical interpretation to the directional argument y . This, in turn, may shed more light on the possibility that the ETUFT - in addition to unifying gravity and electromagnetism - could also describe some micro physical phenomena.

⁶This is easy to see. An electromagnetic field carries energy, that is mass (since energy and mass are equivalent). But mass is the source of gravitational field.

- In conclusion, we note that both Einstein's and Maxwell's equations in the ETUFT are **doubled**. This is due to the splitting of the field equations induced by the nonlinear connection. An **interpretation** of the new equations is needed. Perhaps it is possible to connect one of Maxwell's fields with $SU(2)$ gauge fields. Such a step seems necessary, given the spectacular success of Weinberg-Salam theory. This point will be the subject of future research.

References

- [1] R. S. De Souza and R. Opher, *Origin of $10^{15} - 10^{16} G$ magnetic field in the central engine of gamma ray busts*, JCAP, **02** (2010), 022. (doi: 10.1088/1475-7516/2010/02/022).
- [2] F. I. Mikhail, *Tetrad vector fields and generalizing the theory of relativity*, Ain Shams Sc. Bull., **6** (1962), 87-111.
- [3] F. I. Mikhail and M. I. Wanas, *A generalized field theory. I. Field equations*, Proc. R. Soc. London, A **356** (1977), 471-481.
- [4] F. I. Mikhail and M. I. Wanas, *A generalized field theory. II. Linearized field equations*, Int. J. theoret. phys., **20** (1981), 671- .
- [5] F. I. Mikhail, M. I. Wanas and A. M. Eid, *Theoretical interpretation of cosmic magnetic fields*. Astrphys. Space Sci., **228** (1995), 221-237.
- [6] R. Miron, *Compendium on the geometry of Lagrange spaces*, in Handbook of Differential Geometry, II, 2006, 437-512
- [7] R. Miron and M. Anastasiei, *The geometry of Lagrange spaces: theory and applications*, Kluwer Academic Publishers, 1994.
- [8] R. Miron and M. Anastasiei, *Vector bundles and Lagrange spaces with applications to relativity*, Geometry Balkan Press, Bucurest, Romania, 1997.
- [9] H. P. Robertson, *Groups of motion in spaces admitting absolute parallelism*, Ann. Math, Princeton (2), **33** (1932), 496-520.
- [10] S. Vacaru, *Nonholonomic distribution and gauge models of Einstein gravity*, Int. J. Geom, Meth. Mod. Phys., **7** (2010), 215-246.
- [11] M. I. Wanas, *A generalized field theory and its applications in cosmology*, Ph. D. Thesis, Cairo University, 1975.
- [12] M. I. Wanas, *Absolute parallelism geometry: Developments, applications and problems*, Stud. Cercet, Stiin. Ser. Mat. Univ. Bacau, **10** (2001), 297-309.
- [13] M. I. Wanas, *On the relation between mass and charge: A pure geometric approach*, Int. J. Geom. Meth. Mod. Phys., **4** (2007), 373-388.

- [14] M. I. Wanas, Nabil. L. Youssef and A. M. Sid-Ahmed, *Teleparallel Lagrange geometry and a generalized unified field theory*, Class. Quantum Grav., **27** (2010), 045005 (29pp).
- [15] Nabil. L. Youssef and A. M. Sid-Ahmed, *Linear connections and curvature tensors in the geometry of parallelizabl manifolds*, Rep. Math. Phys., **60** (2007), 39-52.
- [16] Nabil. L. Youssef and A. M. Sid Ahmed, *Extended absolute parallelism geometry*. Int. J. Geom. Meth. Mod. Phys., **5** (2008) 1109-1135.